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CONCERNING THE EXISTENCE AND CHARACTERISTICS OF THE INTEGRAL MULTIFORMITY OF A SYSTEM OF DIFFERENTIAL EQUATIONS WITH A LAGGING ARGUMENT

by V. I. Fodchuk

The existence of an integral multiformity of a system of differential equations with a lagging argument, as indicated in (1), will be proved in this work

$$\frac{dh(t)}{dt} = Hh(t) + H_1 h(t-\Delta) + Q(t, g(t), g(t-\Delta), h(t), h(t-\Delta), \varepsilon),$$
$$\frac{dg(t)}{dt} = \omega + P(t, g(t), g(t-\Delta), h(t), h(t-\Delta), \varepsilon), \tag{1}$$

where h(t), h(t $-\Delta$) are vectors of an n-dimensional Euclidean space E^n, g(t) is a scalar, ω is a constant, Δ is a constant positive lag, and H, H_1 are matrixes of the order of n.

Under study are certain characteristics of that multiformity; in particular, it has been established that an integral multiformity can attract the trajectories of any solutions to system (1), not overlying that multiformity, whose initial functions originate in the sphere of multiformity definition.

With reference to system (1) under consideration, we suggest that the following conditions be satisfied:

1) the function $P(t, g(t), g(t-\Delta), h(t), h(t-\Delta), \varepsilon)$ and the n-dimensional vector-function $Q(t, g(t), g(t-\Delta), h(t), h(t-\Delta), \varepsilon)$ are defined and continuous according to all arguments in the following region

$$t \in (-\infty, \infty),\ g(t),\ g(t-\Delta) \in (-\infty, \infty),\quad h(t),\ h(t-\Delta) \in U_{\rho_0},\ \varepsilon \in (0, \varepsilon_0] \tag{2}$$

and possess a 2π period in relation to the variables g(t), g(t $-\Delta$), where U_{ρ_0} is the region in space E^n in which $|h| < \rho_0$;

2) when $t \in (-\infty, \infty),\ g(t),\ g(t-\Delta) \in (-\infty, \infty),\ h(t) = h(t-\Delta) = 0$, the following inequalitites occur

$$\begin{aligned} |P(t, g(t),\ g(t-\Delta),\ 0,\ 0,\ \varepsilon)| &< M(\varepsilon), \\ |Q(t, g(t),\ g(t-\Delta),\ 0,\ 0,\ \varepsilon)| &\leqslant M(\varepsilon), \end{aligned} \tag{3}$$

where, given $\varepsilon \to 0,\ M(\varepsilon) \to 0$;

3) for any positive $\sigma < \varrho_0$ in the following region

$$\begin{gathered} t \in (-\infty, \infty),\ g(t),\ g(t-\Delta) \in (-\infty, \infty), \\ h(t),\ h(t-\Delta) \in U_\sigma,\ \varepsilon \in [0, \varepsilon_0] \end{gathered}$$

the functions $P(t, g(t),\ g(t-\Delta),\ h(t),\ h(t-\Delta), \varepsilon), Q(t, g(t),\ g(t-\Delta),\ h(t),\ h(t-\Delta), \varepsilon)$ satisfy the Lipshits conditions for $g(t),\ g(t-\Delta),\ h(t),\ h(t-\Delta)$, with a $\lambda(\varepsilon, \sigma) \to 0$ constant at $\varepsilon \to 0,\ \sigma \to 0$;

4) all the real portions of the radicals $p_1, p_2, \ldots, p_n, \ldots$ of the following equation

$$\operatorname{Det}\{Ep - H - H_1 e^{-\Delta p}\} = 0$$

are different from zero and negative, that is

$$\operatorname{Re}(p_k) \leqslant -\gamma_0 < 0 \quad (k = 1, 2, \ldots, n, \ldots),$$

and, consequently, for the matrix $V(t, \tau)$ which satisfies the following equation when $t > \tau$

$$\frac{dV(t, \tau)}{dt} = HV(t, \tau) + H_1 V(t-\Delta, \tau) \tag{4}$$

and the initial conditions of

$$V(t, \tau)|_{t<\tau} \equiv 0,\ V(t, \tau)|_{t=\tau} = E,$$

the following estimate holds true (see [1])

$$|V(t, \tau)| \leq K e^{-\gamma_1 (t-\tau)},\ t \geqslant \tau, \tag{5}$$

where K and γ_1 represent some positive constant, and $\gamma_1 < \gamma_0$, and E is a unit n-dimensional matrix.

The above results represent a generalization of some of N. N. Bogolubov's theorems, outlined in monograph [2], in connection with a system

of differential equations with a lagging argument of type (1). N. N. Bogolubov's method is used to prove the theorems formulated below.

Theorem 1. If the system (1) fulfills the conditions 1 — 4, then there exists such a positive $\bar{\varepsilon}(\bar{\varepsilon} < \varepsilon_0)$, that for every $\varepsilon \in [0, \bar{\varepsilon}]$ that system has an integral multiformity represented by the following correlation

$$h = f(t, g(t), g(t-\Delta), \varepsilon), \tag{6}$$

in which $f(t, g(t), g(t-\Delta), \varepsilon)$, as a function of t, $g(t)$, $g(t-\Delta)$, is defined for $t \in (-\infty, \infty)$, $g(t), g(t-\Delta) \in (-\infty, \infty)$ and fulfills the following inequality

$$\begin{aligned} &|f(t, g(t), g(t-\Delta), \varepsilon)| \leq D(\varepsilon) < \varrho_0, \\ &|f(t, g'(t), g'(t-\Delta), \varepsilon) - f(t, g''(t), g''(t-\Delta), \varepsilon)| \leq \\ &\leq \gamma(\varepsilon)\{|g'(t) - g''(t)| + |g'(t-\Delta) - g''(t-\Delta)|\}, \end{aligned} \tag{7}$$

with $D(\varepsilon) \to 0$, $\gamma(\varepsilon) \to 0$ and $\varepsilon \to 0$. The function $f(t, g(t), g(t-\Delta), \varepsilon)$ has a period of 2 in relations to the variables $g(t)$, $g(t-\Delta)$.

We shall present a brief outline of the basic ideas only behind the proof of Theorem 1.

Let us take some class $C(D, \gamma)$ of functions $F[t, g(t), g(t-\Delta), \varepsilon]$, where D, γ are fixed positive numbers with E^n values, defined for $t \in (-\infty, \infty)$, $g(t)$, $g(t-\Delta) \in (-\infty, \infty)$ fulfilling the following inequalities

$$\begin{aligned} &|F[t, g(t), g(t-\Delta)]| \leq D, \\ &|F[t, g'(t), g'(t-\Delta)] - F[t, g''(t), g''(t-\Delta)]| < \\ &< \gamma\{|g'(t) - g''(t)| + |g'(t-\Delta) - g''(t-\Delta)|\} \end{aligned} \tag{8}$$

and possessing a period of 2π in relations to the variables $g(t)$, $g(t-\Delta)$.

Let us take some function of $F[t, g(t), g(t-\Delta)]$ from class $C(D, \gamma)$ and consider the following equation for it

$$\begin{aligned} \frac{dg(t)}{dt} = \omega + P(t, g(t), g(t-\Delta), F[t, g(t), g(t-\Delta)], F[t- \\ -\Delta, g(t-\Delta), g(t-2\Delta)], \varepsilon) \end{aligned} \tag{9}$$

Setting ourselves the arbitrary initial values $g(t) = g_0(t)$ for $t \in [t_0 - \Delta, t_0]$, where $g_0(t)$ is taken from some class of continuous functions of C^0, we shall construct the solution of equation (9) for $t \geqslant t_0$, and it

will be the only solution. We shall designate it symbolically as $g_t = T^F_{z,t_0}(g_0(t))$, where $z = t - t_0$.

We shall now see how S of function $F[t, g(t), g(t-\Delta)]$ is transformed from class $C(D,\gamma)$ into the following function, as in Yu. A. Mitorpol'skiy's work [3],

$$S_{t,g}(F) = \int_{-\infty}^{0} V(t, t+z)\, Q\{t+z;\ T^F_{z,t}(g(t));\ T^F_{z+\Delta,t-\Delta}(g(t));$$
$$F[t+z;\ T^F_{z,t}(g(t));\ T^F_{z+\Delta,t-\Delta}(g(t))];\ F[t+z-\Delta;\ T^F_{z+\Delta,t-\Delta}(g(t));$$
$$T^F_{z+2\Delta,t-2\Delta}(g(t))];\ \varepsilon\}\, dt. \tag{10}$$

Next we will prove that transformation (10) represents a class of functions of $C(D,\gamma)$, and that the following inequality occurs

$$|SF^* - SF| < \frac{1}{q}|F^* - F|, \qquad q > 1,$$

thereby proving the existence of the only solution of equality

$$SF = F, \tag{11}$$

which we shall designate as $f(t, g(t), g(t-\Delta), \varepsilon)$. The function of $f(t, g(t), g(t-\Delta), \varepsilon)$ will be periodic (with a period of 2π), according to $g(t)$ and $g(t-\Delta)$.

To complete the proof, all that has to be shown is that the correlation

$$h = f(t, g(t), g(t-\Delta), \varepsilon) \tag{12}$$

defines the integral multiformity for system (1).

To that end, we shall substitute function (12) for F in equation (11), replace $g(t)$ by $T^f_{z,t}(g(t))$ in the resulting identity and introduce a new integration variable $\tau = z + t$. Then assuming for the purpose of brevity that

$$f[t, T^I_{t-t_0,t_0}(g(t)),\ T^I_{t-t_0+\Delta,t_0-\Delta}(g(t)), \varepsilon] = h_t,$$
$$T^I_{t-t_0,t_0}(g(t)) = g_t, \tag{13}$$

we get the following identity

$$h_t = \int_{-\infty}^{t} V(t,\tau)\, Q(\tau, g_\tau, g_{\tau-\Delta}, h_\tau, h_{\tau-\Delta}, \varepsilon)\, d\tau. \tag{14}$$

Differentiating it by t, as a parameter, we find

$$\frac{dh_t}{dt} = \int_{-\infty}^{t} \frac{dV(t,\tau)}{dt} Q(\tau, g_\tau, g_{\tau-\Delta}, h_\tau, h_{\tau-\Delta}, \varepsilon)\, d\tau + \\ + Q(t, g_t, g_{t-\Delta}, h_t, h_{t-\Delta}, \varepsilon). \tag{15}$$

Hence, according to (4) and (14), we get the final result

$$\frac{dh_t}{dt} = Hh_t + H_1 h_{t-\Delta} + Q(t, g_t, g_{t-\Delta}, h_t, h_{t-\Delta}, \varepsilon).$$

We should point out that g_t, h_t fulfill the second equation of system (1) by the very definition of the functional $T^f_{t-t_o, t_n}$.

Consequently, g_t and h_t, defined by formulas (13), are the solutions of system (1), and expression (12) represents the integral multiformity of system (1).

Theorem 2. If the functions $P(t, g(t), g(t-\Delta), h(t), h(t-\Delta), \mathcal{E})$ and $Q(t, g(t), g(t-\Delta), h(t), h(t-\Delta))$, with $0 < \mathcal{E} < \mathcal{E}_o$, have limited and equicontinuous derivatives g and h up to the m-th order inclusive, then $f(t, g(t), h(t-\Delta), \mathcal{E})$ will also have limited and equicontinuous derivatives up to the m-th order inclusive.

Theorem 3. If there exists a sequence of numbers (τ_m) such that the right-hand parts of equations (1), with reference to some $\varepsilon < \bar{\varepsilon}$, fulfill the correlations shown below uniformly for $t \in (-\infty, \infty)$, $g(t)$, $g(t-\Delta)$ $(-\infty, \infty)$, $h(t)$, $h(t-\Delta) \in U_Q$,

$$|P(t+\tau_m, g(t), g(t-\Delta), h(t), h(t-\Delta), \varepsilon) - \\ - P(t, g(t), g(t-\Delta), h(t), h(t-\Delta), \varepsilon)| \underset{m\to\infty}{\to} 0,$$

$$|Q(t+\tau_m g(t), g(t-\Delta), h(t), h(t-\Delta), \varepsilon) - \\ - Q(t, g(t), g(t-\Delta), h(t), h(t-\Delta), \varepsilon)| \underset{m\to\infty}{\to} 0,$$

then $\mathcal{E}$ is uniform for $t \in (-\infty, \infty)$, $g(t)$, $g(t-\Delta) \in (-\infty, \infty)$ and the following correlation is valid

$$|f(t+\tau_m, g(t), g(t-\Delta), \varepsilon) - f(t, g(t), g(t-\Delta), \varepsilon)| \underset{m\to\infty}{\to} 0,$$

where $f(t,g(t), g(t-\Delta), \mathcal{E})$ is the integral multiformity for system (1).

Corollary. If the functions $P(t,g(t), g(t-\Delta), h(t), h(t-\Delta), \mathcal{E})$ and $Q(t,g(t), g(t-\Delta), h(t), h(t-\Delta), \mathcal{E})$ are almost periodic functions of t, uniform in relation to $g(t)$, $g(t-\Delta)$, $h(t)$, $h(t-\Delta)$, for $t \in (-\infty,\infty)$, $g(t)$, $g(t-\Delta) \in (-\infty,\infty)$, $h(t)$, $h(t-\Delta) \in U_{Q_0}$, the function $f(t,g(t), g(t-\Delta), \mathcal{E})$ will be an almost periodic function, uniform in relation to $g(t)$, $g(t-\Delta) \in (-\infty,\infty)$ with a frequency base of functions P & Q.

Theorem 4. It is possible to indicate such positive constants $\mathcal{E}_1$, ρ_1, $(\mathcal{E}_1 \leqslant \bar{\mathcal{E}} < \mathcal{E}_0, \rho_1 < \rho_0)$ that for every $\mathcal{E} \in [0, \mathcal{E}_1]$ and arbitrary real t_0, any solution h_t of system (1) fulfilling the initial conditions is

$$h_t = \varphi_0(t,\varepsilon) \quad \text{for} \quad t \in [t_0-\Delta, t_0], \qquad |\varphi_0(t,\varepsilon)| \leqslant \rho_1, \tag{16}$$

where $\varphi_0(t,\mathcal{E})$, a continuous initial function, will in the course of time tend asymptotically to a single integral multiformity whose parametric representation we shall designate as $f(t,g(t), g(t-\Delta), \mathcal{E})$ which is defined and limited for all $t \in (-\infty,\infty)$.

Remark. In proving theorem 4, we substitute a system of integral-differential equations for systems (1) and initial conditions (16), as is done in work [4].

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